Multiple-Antenna Interference Channel with Receive Antenna Joint Processing and Real Interference Alignment

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***Abstract*—We consider a constant K-user Gaussian interference channel with M antennas at each transmitter and N antennas at each receiver, denoted as a (K, M, N) channel. Relying on a result on simultaneous Diophantine approximation, a real interference alignment scheme with joint receive antenna processing is developed. The scheme is used to provide new proofs for two previously known results, namely 1) the total degrees of freedom (DoF) of a (K, N, N) channel is $NK/2$; and 2) the total DoF of a (K, M, N) channel is at least $KMN/(M+N)$. We also derive the DoF region of the (K, N, N) channel, and an inner bound on the DoF region of the (K, M, N) channel.**

I. Introduction

Interference channel is an important model for multi-user communication systems. In a K-user interference channel, the k-th transmitter has a message intended for the k-th receiver. At receiver k, the messages from transmitters other than the k-th are interference. Characterizing the capacity region of a general interference channel is an open problem, although results for some specific cases are known.

To quantify the shape of the capacity region at high signal-to-noise ratio (SNR), the concept of degrees of freedom (DoF) has been introduced [1], [2]. The DoF of a message is its rate normalized by the capacity of single-user additive white Gaussian noise channel, as the SNR tends to infinity.

To achieve the optimal DoF, the concept of interference alignment turns out to be important [3]. At a receiver, the interference signals from multiple transmitters are aligned in the signal space, so that the dimensionality of the interference in the signal space can be minimized. Therefore, the remaining space is interference free and can be used for the desired signals. Two commonly used alignment schemes are vector alignment and real alignment. In real alignment, the concept of linear independence over the rational numbers replaces the more familiar vector linear independence. And a Groshev type of theorem is usually used to guarantee the required decoding performance [4].

So far the real alignment schemes have been mainly developed only for scalar interference channels. For multiple-input multiple output (MIMO) interference channels, antenna splitting argument has been used in [4] and [5] to derive the total DoF. In such antenna splitting arguments, no cooperation is employed either at the transmitter side or at the receiver side.

In this paper, we consider a constant K-user Gaussian interference channel with M antennas at each transmitter and N antennas at each receiver, denoted as a (K, M, N) *channel*. We develop a real alignment scheme for MIMO interference channel that employs joint receive antenna processing. Relying on the recent results on simultaneous Diophantine approximation, we are able to obtain new proofs of two previously known results, and derive two new results on the DoF region; see Sec. III.

II. System model

Notation: K, D, D', and N are integers and $\mathcal{K} = \{1, \ldots, K\}$, $\mathcal{N} = \{1, \ldots, N\}$. We use k and $\hat{k}$ as transmitter indices, and j as receiver indices. Superscripts t and r are used for transmitter and receiver antenna indices. The set of integers and real numbers are denoted as $\mathbb{Z}$ and $\mathbb{R}$, respectively. The set of non-negative real numbers is denoted as $\mathbb{R}_+$. Letter i and l are used as the indices of directions and streams, respectively. Vectors and matrices are indicated by bold symbols. We use $\|\mathbf{x}\|$ to denote infinity norm of $\mathbf{x}$, $(\cdot)^*$ matrix transpose, and $\otimes$ the Kronecker product of two matrices.

Consider a multiple-antenna K-user real Gaussian interference channel with M antennas at each transmitter and N antennas at each receiver. At each time, each transmitter, say transmitter k, sends a vector signal $\mathbf{x}_k \in \mathbb{R}^M$ intended for receiver k. The channel from transmitter k to receiver j is represented as a matrix

$$\mathbf{H}_{j,k} := [h_{j,k,r,t}]_{r=1,t=1}^{N,M} \tag{1}$$

where $k \in \mathcal{K}$, $j \in \mathcal{K}$, and $\mathbf{H}_{j,k} \in \mathbb{R}^{N\times M}$. It is assumed that the channel is constant during all transmissions. Each transmitter is subjected to a power constraint P. The received signal at receiver j can be expressed as

$$\mathbf{y}_j = \sum_{k\in\mathcal{K}} \mathbf{H}_{j,k}\mathbf{x}_k + \boldsymbol{\nu}_j, \quad \forall j \in \mathcal{K} \tag{2}$$

where $\{\boldsymbol{\nu}_j | j \in \mathcal{K}\}$ is the set of independent Gaussian additive noises with real, zero mean, independent, and unit variance entries. Let $\mathbf{H}$ denote the $(KN) \times (KM)$ block matrix, whose (j, k)th block of size $N \times M$ is the matrix $\mathbf{H}_{j,k}$. The matrix $\mathbf{H}$ includes all the channel coefficients. For the

interference channel $\mathbf{H}$, the *capacity region* $\mathcal{C}(P, K, \mathbf{H})$ is defined in the usual sense: It contains rate tuples $\mathbf{R}_K(P) = [R_1(P), R_2(P), \ldots, R_K(P)]$ such that reliable transmission from transmitter k to receiver k is possible at rate R_k for all $k \in \mathcal{K}$ simultaneously, under the given power constraint P. Reliable transmissions mean that the probability of error can be made arbitrarily small by increasing the encoding block length while keeping the rates and power fixed.

A DoF vector $\mathbf{d} = (d_1, d_2, \ldots, d_K)$ is said to be *achievable* if for any large enough P, the rates $R_i = 0.5 \log(P) d_i$, $i = 1, 2, \ldots, K$, are simultaneously achievable by all K users, namely $0.5 \log(P) \cdot \mathbf{d} \in \mathcal{C}(P, K, \mathbf{H})$, for P large enough. The *DoF region* for a given channel $\mathbf{H}$, $\mathcal{D}(K, \mathbf{H})$, is the closure of the set of all achievable DoF vectors. The DoF region $\mathcal{D}(K, M, N)$ is the largest possible region such that $\mathcal{D}(K, M, N) \subset \mathcal{D}(K, \mathbf{H})$ for almost all $\mathbf{H}$ in the Lebesgue sense. The *total DoF of the K-user interference channel* $\mathbf{H}$ is defined as

$$d(K, \mathbf{H}) = \max_{\mathbf{d} \in \mathcal{D}(K, \mathbf{H})} \sum_{k=1}^{K} d_k.$$

The *total DoF* $d(K, M, N)$ is defined as the largest possible real number μ such that for almost all (in the Lebesgue sense) real channel matrices $\mathbf{H}$ of size $(KN) \times (KM)$, $d(K, \mathbf{H}) \geq \mu$.

III. Main Results

The following two theorems have been proved before, in [4] and [5], respectively:

Theorem 1: $d(K, N, N) = \frac{NK}{2}$.

Theorem 2: $d(K, M, N) \geq \frac{MN}{M+N} K$.

The main contributions of the paper are 1) providing alternative proofs of the above two theorems, and 2) prove the following theorems.

Theorem 3: The DoF region of a (K, N, N) interference channel is the following

$$\mathcal{D}(K, N, N) = \{\mathbf{d} \in \mathbb{R}_+^{K \times 1} \,\big|\, d_k + \max_{\hat{k} \neq k} d_{\hat{k}} \leq N, \forall k \in \mathcal{K}\}.$$

Theorem 4: The DoF region of a (K, M, N) interference channel satisfies $\mathcal{D}(K, N, N) \supset \mathcal{D}^{(\text{in})}$ where

$$\mathcal{D}^{(\text{in})} \coloneqq \{\mathbf{d} \in \mathbb{R}_+^{K \times 1} \,\big|\, M d_k + N \max_{\hat{k} \neq k} d_{\hat{k}} \leq MN, \quad \forall k \in \mathcal{K}\}.$$

Remark 1: The DoF region of K-user time-varying interference channel with N antennas at each node has been obtained before in [6]. The fact the DoF region of a (K, N, N) channel in Theorem 3 is the same as that of a time-varying MIMO interference channel with the same number of antennas indicates that the DoF region for this channel is an inherent spatial property of the channel that is separate from the time or frequency diversity.

Remark 2: Theorem 3 follows from Theorem 4 by setting $M = N$, and the Multiple Access Channel (MAC) outer bound [7, Sec. 14.3].

Remark 3: Theorem 2 follows from Theorem 4 by setting $d_k = MN/(M+N)$, $\forall k \in \mathcal{K}$.

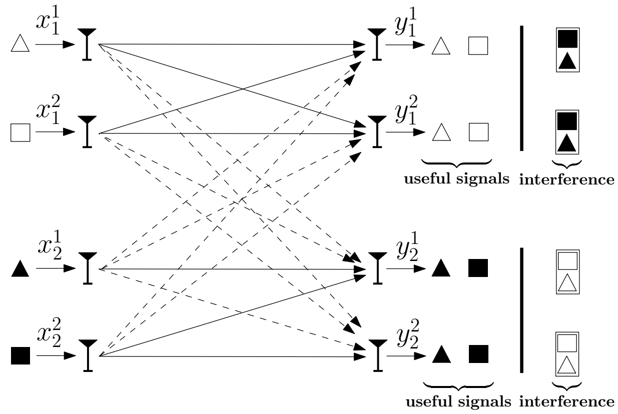

Fig. 1. 2-user Gaussian interference channel with 2 antennas at each transmitter receiver

Remark 4: Theorem 1 follows from Theorem 2 by setting $M = N$ and the outer bound for K-user interference channel that has been obtained before in [2].

From the above remarks, it is only necessary to prove Theorem 4. However, we will first prove the achievability of Theorem 1 in the next section, which serves to introduce the joint antenna processing at the receivers, and the application of the result in simultaneous Diophantine approximation on manifolds. Theorem 4 will be proved in Sec. V.

IV. Achievability Proof for Theorem 1

One important technique for proving achievability result is the real interference alignment [4] which seeks to align the dimensions of interferences so that more free dimensions can be available for intended signals. The dimensions (also named directions) are represented as real numbers that are rationally independent (cf. Appendix A).

We will denote set of directions, a specific direction, and vector of directions using $\mathcal{T}$, T, and $\mathbf{T}$ respectively.

Encoding: Transmitter k sends a vector message $\mathbf{x}_k = (x_k^1, \ldots, x_k^N)^*$ where x_k^t, $\forall t \in \mathcal{N}$ is the signal emitted by antenna t at transmitter k. The signal x_k^t is generated using transmit directions in a set $\mathcal{T} = \{T_i \in \mathbb{R} | 1 \leq i \leq D\}$ as follows

$$x_k^t = \mathbf{T} \mathbf{s}_k^t \tag{3}$$

where

$$\mathbf{T} \coloneqq (T_1, \ldots, T_D), \quad \mathbf{s}_k^t \coloneqq (s_{k1}^t, \ldots, s_{kD}^t)^* \tag{4}$$

and $\forall 1 \leq i \leq D$,

$$s_{ki}^t \in \{\lambda q \,\big|\, q \in \mathbb{Z}, -Q \leq q \leq Q\}. \tag{5}$$

The parameters Q and λ will be designed to satisfy the rate and power constraints.

Alignment Design: We design transmit directions in such a way that at any receiver antenna, each useful signal occupies a set of directions that are rationally independent of interference directions (cf. Appendix A).

To make it more clear, consider Fig. 1. x_1^1 and x_1^2 are shown by white triangle and square. In a similar fashion, x_2^1 and x_2^2 are indicated with black triangle and square. We are interested in such transmit directions that at each receiver antenna the interferences, for instance black triangle and square at receiver 1, are aligned while the useful messages, white triangle and square, occupy different set of directions.

TRANSMIT DIRECTIONS: Our scheme requires all transmitter antennas to only contain directions of the following form

$$T = \prod_{j \in \mathcal{K}} \prod_{k \in \mathcal{K}, k \neq j} \prod_{r \in \mathcal{N}} \prod_{t \in \mathcal{N}} (h_{j,k,r,t})^{\alpha_{j,k,r,t}} \tag{6}$$

where

$$0 \leq \alpha_{j,k,r,t} \leq n-1, \tag{7}$$

$\forall j \in \mathcal{K}$, $k \in \mathcal{K}$, $k \neq j$, $r \in \mathcal{N}$, $t \in \mathcal{N}$. It is easy to see that the total number directions is

$$D = n^{K(K-1)N^2}. \tag{8}$$

We also assume that directions in $\mathcal{T}$ are indexed from 1 to D. The exact indexing order is not important here.

ALIGNMENT ANALYSIS: Our design proposes that at each antenna of receiver j, $j \in \mathcal{K}$, the set of messages $\{x_k^t | k \in \mathcal{K}, k \neq j, t \in \mathcal{N}\}$ are aligned. To verify, consider all x_k^t, $k \neq j$ that are generated in directions of set $\mathcal{T}$. These symbols are interpreted as the interferences for receiver j. Let

$$D' = (n+1)^{K(K-1)N^2}. \tag{9}$$

and define a set $\mathcal{T}' = \{T_i' \in \mathbb{R} | 1 \leq i \leq D'\}$ such that all T_i' are in from of T as in (6) but with a small change as follows

$$0 \leq \alpha_{j,k,r,t} \leq n. \tag{10}$$

Clearly, all x_k^t, $k \neq j$ arrive at antenna r of receiver j in the directions of $\{(h_{j,k,r,t})\, T | k \in \mathcal{K}, k \neq j, t \in \mathcal{N}, T \in \mathcal{T}\}$ which is a subset of $\mathcal{T}'$.

This confirms that at each antenna of any receiver, all the interferences only contain the directions from $\mathcal{T}'$. These interference directions can be described by a vector

$$\mathbf{T}' \coloneqq (T_1', \ldots, T_{D'}'). \tag{11}$$

DECODING SCHEME: In this part, we first rewrite the received signals. Then, we prove the achievability part of Theorem 1 using joint antenna processing.

The received signal at receiver j is represented by

$$\mathbf{y}_j = \mathbf{H}_{j,j}\mathbf{x}_j + \sum_{k \in \mathcal{K}, k \neq j} \mathbf{H}_{j,k}\mathbf{x}_k + \boldsymbol{\nu}_j. \tag{12}$$

Let us define

$$\mathbf{B} \coloneqq \begin{pmatrix} \mathbf{T} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{T} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{T} \end{pmatrix} \tag{13}$$

and

$$\mathbf{s}_k \coloneqq \begin{pmatrix} \mathbf{s}_k^1 \\ \mathbf{s}_k^2 \\ \vdots \\ \mathbf{s}_k^N \end{pmatrix}, \quad \mathbf{u}_k = \frac{\mathbf{s}_k}{\lambda}, \tag{14}$$

such that $\mathbf{B}$ is a $N \times ND$ matrix with $(N-1)D$ zeros at each row. Using above definitions, $\mathbf{y}_j$ can be rewritten as

$$\mathbf{y}_j = \lambda \left(\mathbf{H}_{j,j}\mathbf{B}\mathbf{u}_j + \sum_{k \in \mathcal{K}, k \neq j} \mathbf{H}_{j,k}\mathbf{B}\mathbf{u}_k \right) + \boldsymbol{\nu}_j. \tag{15}$$

The elements of $\mathbf{u}_k$ are integers between $-Q$ and Q, cf. (5).

We rewrite

$$\mathbf{H}_{j,j}\mathbf{B}\mathbf{u}_j = (\mathbf{H}_{j,j} \otimes \mathbf{T})\,\mathbf{u}_j = \begin{pmatrix} h_{j,j,1,1}\mathbf{T} & h_{j,j,1,2}\mathbf{T} & \ldots & h_{j,j,1,N}\mathbf{T} \\ h_{j,j,2,1}\mathbf{T} & h_{j,j,2,2}\mathbf{T} & \ldots & h_{j,j,2,N}\mathbf{T} \\ \vdots & \vdots & \ddots & \vdots \\ h_{j,j,N,1}\mathbf{T} & h_{j,j,N,2}\mathbf{T} & \ldots & h_{j,j,N,N}\mathbf{T} \end{pmatrix} \mathbf{u}_j \coloneqq \begin{pmatrix} \mathbf{T}_j^1 \\ \mathbf{T}_j^2 \\ \vdots \\ \mathbf{T}_j^N \end{pmatrix} \mathbf{u}_j \tag{16}$$

where $\forall r \in \mathcal{N}$, $\mathbf{T}_j^r$ is the r^{th} row of $\mathbf{H}_{j,j}\mathbf{B}$. Also,

$$\sum_{k \in \mathcal{K}, k \neq j} \mathbf{H}_{j,k}\mathbf{B}\mathbf{u}_k = \sum_{k \in \mathcal{K}, k \neq j} (\mathbf{H}_{j,k} \otimes \mathbf{T})\,\mathbf{u}_k = \begin{pmatrix} \sum_{k \in \mathcal{K}, k \neq j} \sum_{t \in \mathcal{N}} (h_{j,k,1,t}\mathbf{T}\mathbf{u}_k^t) \\ \sum_{k \in \mathcal{K}, k \neq j} \sum_{t \in \mathcal{N}} (h_{j,k,2,t}\mathbf{T}\mathbf{u}_k^t) \\ \vdots \\ \sum_{k \in \mathcal{K}, k \neq j} \sum_{t \in \mathcal{N}} (h_{j,k,N,t}\mathbf{T}\mathbf{u}_k^t) \end{pmatrix} \stackrel{(a)}{=} \begin{pmatrix} \mathbf{T}'\mathbf{u}_j'^1 \\ \mathbf{T}'\mathbf{u}_j'^2 \\ \vdots \\ \mathbf{T}'\mathbf{u}_j'^N \end{pmatrix} \tag{17}$$

where $\forall r \in \mathcal{N}$, $\mathbf{u}_j'^r$ is a column vector with D' integer elements, and (a) follows since the set $\mathcal{T}'$ contains all directions of the form $(h_{j,k,r,t})\, T$ where $k \neq j$; cf. the definition of $\mathcal{T}'$.

Considering (16) and (17), we are able to equivalently denote $\mathbf{y}_j$ as

$$\mathbf{y}_j = \lambda \begin{pmatrix} \mathbf{T}_j^1 & \mathbf{T}' & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{T}_j^2 & \mathbf{0} & \mathbf{T}' & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_j^N & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{T}' \end{pmatrix} \begin{pmatrix} \mathbf{u}_j \\ \mathbf{u}_j'^1 \\ \vdots \\ \mathbf{u}_j'^N \end{pmatrix} + \boldsymbol{\nu}_j. \tag{18}$$

We finally left multiply $\mathbf{y}_j$ by an $N \times N$ weighting matrix

$$\mathbf{W} = \begin{pmatrix} 1 & \gamma_{12} & \ldots & \gamma_{1N} \\ \gamma_{21} & 1 & \ldots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \ldots & 1 \end{pmatrix} \tag{19}$$

such that all indexed γ are randomly, independently, and uniformly chosen from interval $[\frac{1}{2}, 1]$. This process causes the zeros in (18) to be filled by non-zero directions.

After multiplying $\mathbf{W}$, the noiseless received constellation belongs to a lattice generated by the $N \times N(D+D')$ matrix

$$\mathbf{A} = \mathbf{W} \begin{pmatrix} \mathbf{T}_j^1 & \mathbf{T}' & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{T}_j^2 & \mathbf{0} & \mathbf{T}' & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_j^N & \mathbf{0} & \mathbf{0} & \dots & \mathbf{T}' \end{pmatrix}. \tag{20}$$

The above matrix has a significant property that allows us to use Theorem 5 (cf. Appendix C). Theorem 5 requires each row of $\mathbf{A}$ to be a nondegenerate map from a subset of channel coefficients to $\mathbb{R}^{N(D+D')}$. The nondegeneracy is established because (cf. Appendix B):

1) all elements of $\mathbf{T}'$ and $\mathbf{T}_j^t, \forall t \in \mathcal{N}$ are analytic functions of the channel coefficients;
2) all the directions in $\mathbf{T}'$ and $\mathbf{T}_j^t$, $\forall t \in \mathcal{N}$ together with 1 are linearly independent over $\mathbb{R}$;
3) all indexed γ in $\mathbf{W}$ have been chosen randomly and independently.

Hence, using Theorem 5, the set of $\mathbf{H}$ such that there exist infinitely many

$$\mathbf{q} = \begin{pmatrix} \mathbf{u}_j \\ \mathbf{u}_j'^1 \\ \vdots \\ \mathbf{u}_j'^N \end{pmatrix} \in \mathbb{Z}^{N(D+D')}$$

with

$$\|\mathbf{A}\mathbf{q}\| < \|\mathbf{q}\|^{-(D+D')-\epsilon} \quad \text{for } \epsilon > 0 \tag{21}$$

has zero Lebesgue measure. In other words, for almost all $\mathbf{H}$, $\|\mathbf{A}\mathbf{q}\| > \|\mathbf{q}\|^{-(D+D')-\epsilon}$ holds for all $\mathbf{q} \in \mathbb{Z}^{N(D+D')}$ except for finite number of them. By the construction of $\mathbf{A}$, all elements in each row of $\mathbf{A}$ are rationally independent with probability one, which means that $\mathbf{A}\mathbf{q} \neq \mathbf{0}$ unless $\mathbf{q} = \mathbf{0}$. Therefore, almost surely for any fixed channel (hence fixed $\mathbf{A}$), there is a positive constant β such that $\|\mathbf{A}\mathbf{q}\| > \beta\|\mathbf{q}\|^{-(D+D')-\epsilon}$ holds for all integer $\mathbf{q} \neq \mathbf{0}$. Since

$$\|\mathbf{q}\| \leq (K-1)NQ,$$

the distance between any two points of received constellation (without considering noise) is lower bounded by

$$\beta\lambda\big((K-1)NQ\big)^{-(D+D')-\epsilon}. \tag{22}$$

Remark 5: The noiseless received signal belongs to a constellation of the form

$$\mathbf{y} = \lambda\mathbf{A}\bar{\mathbf{q}} \tag{23}$$

where $\bar{\mathbf{q}}$ is an integer vector. Then, the hard decision maps the received signal to the nearest point in the constellation. Note that the hard decoder employs all N antennas of receiver j to detect signals emitted by intended transmitter. In other words, our decoding scheme is based on multi-antenna joint processing.

We now design the parameters λ and Q. With reference to [4], if we choose

$$\lambda = \zeta\frac{P^{\frac{1}{2}}}{Q}, \tag{24}$$

then the power constraint is satisfied (ζ is a constant here). Moreover, similar to [4], we choose

$$Q = P^{\frac{1-\epsilon}{2(D+D'+1+\epsilon)}} \quad \text{for } \epsilon \in (0,1) \tag{25}$$

assuring that the DoF per direction is $\frac{1-\epsilon}{D+D'+1+\epsilon}$. Since, we are allowed to arbitrarily choose ϵ within $(0,1)$, $\frac{1}{D+D'+1}$ is also achievable.

Using (22)–(25) and the performance analysis described by [4], the hard decoding error probability of received constellation goes to zero as $P \to \infty$ and the total achievable DoF for almost all channel coefficients in the Lebesgue sense is

$$\frac{NKD}{D+D'+1} = \frac{NKn^{K(K-1)N^2}}{n^{K(K-1)N^2} + (n+1)^{K(K-1)N^2} + 1} \tag{26}$$

and as n increases, the total DoF goes to $\frac{NK}{2}$ which meets the outer bound [2].

V. Proof of Theorem 4

Notation: Unless otherwise stated, all the assumptions and definitions are still the same.

Consider the case where the number of transmitter and receiver antennas are not equal. This can be termed the K-user MIMO interference channel with M antennas at each transmitter and N antennas at each receiver. Hence, for all $j \in \mathcal{K}$ and $K \in \mathcal{K}$, $\mathbf{H}_{j,k}$ is a $N \times M$ matrix. We prove that for any $\mathbf{d} \in \mathcal{D}^{(\text{in})}$, $\mathbf{d}$ is achievable.

Under the rational assumption, it is possible to find an integer ρ such that $\forall k \in \mathcal{K}$, $\bar{d}_k = \rho\frac{d_k}{M}$ where $\bar{d}_k$ is a non-negative integer. The signal x_k^t is divided into $\bar{d}_k$ streams. For stream l, $l \in \{1, \dots, \max_{k\in\mathcal{K}} \bar{d}_k\}$, we use directions $\{T_{l1}, \dots, T_{lD}\}$ of the following form

$$T_l = \prod_{j\in\mathcal{K}} \prod_{k\in\mathcal{K}, k\neq j} \prod_{r\in\mathcal{N}} \prod_{t\in\mathcal{N}} (h_{j,k,r,t}\delta_l)^{\alpha_{j,k,r,t}} \tag{27}$$

where $0 \leq \alpha_{j,k,r,t} \leq n-1$ and δ_l is a design parameter that is chosen randomly, independently, and uniformly from the interval $[\frac{1}{2}, 1]$. Let $\mathbf{T}_l := (T_{l1}, \dots, T_{lD})$. Note that, at any antenna of transmitter k, the constants $\{\delta_l\}$ cause the streams to be placed in $\bar{d}_k$ different sets of directions. The alignment scheme is the same as before, considering the fact that at each antenna of receiver j, the useful streams occupy $M\bar{d}_j$ separate sets of directions. The interferences are also aligned at most in $\max_{k\in\mathcal{K},k\neq j} \bar{d}_k$ sets of directions independent from useful directions.

By design, x_k^t is emitted in the following form

$$x_k^t = \sum_{l=1}^{\bar{d}_k} \delta_l \sum_{i=1}^{D} T_{li} s_{kli}^t = \mathbf{T}_k \mathbf{s}_k^t \tag{28}$$

where

$$\mathbf{T}_k := (\delta_1\mathbf{T}_1, \dots, \delta_{\bar{d}_k}\mathbf{T}_{\delta_{\bar{d}_k}}), \tag{29}$$

$$\mathbf{s}_k^t := (s_{k11}^t, \dots, s_{k\bar{d}_k D}^t)^*, \tag{30}$$

and all s_{kli}^t belong to the set defined in (5).

Pursuing the same steps of the previous section for receiver j, $\mathbf{B}$ becomes a $M \times MD\bar{d}_j$ matrix and $\mathbf{A}$ will have N rows and $MD\bar{d}_j + ND' \max_{k\in\mathcal{K},k\neq j} \bar{d}_k$ columns. The total number directions G_j of both useful signals and the interferences at receiver j satisfies

$$G_j \leq MD\bar{d}_j + ND' \max_{k\in\mathcal{K},k\neq j} \bar{d}_k. \tag{31}$$

For any DoF points in $\mathcal{D}^{(\text{in})}$ satisfying Theorem 4, we have

$$G_j \leq \left(M\bar{d}_j + N \max_{k\in\mathcal{K},k\neq j} \bar{d}_k\right) D' \leq \frac{\rho}{M} NMD' = \rho N D' \tag{32}$$

and as n increases, the DoF of the signal x_j intended for receiver j, $\forall j \in \mathcal{K}$ can be arbitrarily close to

$$\lim_{n\to\infty} MD\bar{d}_j \frac{N}{\rho N D'} = \lim_{n\to\infty} \frac{M}{\rho} \frac{\bar{d}_j n^{K(K-1)N^2}}{(n+1)^{K(K-1)N^2}} = \frac{M}{\rho}\bar{d}_j = d_j \tag{33}$$

where $\frac{N}{\rho N D'}$ is the DoF per direction for large D'. This proves Theorem 4.

As a special case, it is easy to see when all d_k are equal, the total achievable DoF is $\frac{MN}{M+N}K$. Moreover, when $M = N$, the achievable DoF region meets the outer bound [2].

VI. Conclusions and Future Works

We developed a new real interference alignment scheme for multiple-antenna interference channel that employs joint receiver antenna processing. The scheme utilized a result on simultaneous Diophantine approximation and aligned all interferences at each receive antenna. We were able to provide new proofs for two existing results on the total DoF of multiple antenna interference channels (Theorem 1 and Theorem 2) and drive two new DoF region results (Theorem 3 and Theorem 4).

It is desired to extend the result of the paper to a multiple-antenna interference network with K transmitters and J receivers where each transmitter sends an arbitrary number of messages, and each receiver may be interested in an arbitrary subset of the transmitted messages. This channel is known as wireless X network with general message demands.

Acknowledgment: The authors thank V. Beresnevich for discussion on the convergence problem of Diophantine approximation on manifolds and directing us to reference [8].

Appendix

A. Definitions of Independence

A set of real numbers is *rationally independent* if none of the elements of the set can be written as a linear combination of the other elements with rational coefficients.

A set of functions are *linearly independent over* $\mathbb{R}$ if none of the functions can be represented by a linear combination of the other functions with real coefficients.

B. Nondegenerate Manifolds [9]

Consider a d-dimensional sub-manifold $\mathcal{M} = \{\mathbf{f}(\mathbf{x}) | \mathbf{x} \in U\}$ of $\mathbb{R}^n$, where $U \subset \mathbb{R}^d$ is an open set and $\mathbf{f} = (f_1, \ldots, f_n)$ is a C^k embedding of U to $\mathbb{R}^n$. For $l \leq k$, $\mathbf{f}(\mathbf{x})$ is an l-*nondegenerate* point of $\mathcal{M}$ if partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ of order up to l span the space $\mathbb{R}^n$. The function $\mathbf{f}$ at $\mathbf{x}$ is *nondegenerate* when it is l-nondegenerate at $\mathbf{x}$ for some l.

If the functions $f_1, \ldots, f_n$ are analytic, and $1, f_1, \ldots, f_n$ are linearly independent over $\mathbb{R}$ in a domain U, all points of $\mathcal{M} = \mathbf{f}(U)$ are nondegenerate.

C. Diophantine approximation for systems of linear forms [8]

Consider a $m \times n$ real matrix $\mathbf{A}$ and $\mathbf{q} \in \mathbb{Z}^n$. The theory of simultaneous Diophantine approximation tries to figure out how small the distance from $\mathbf{Aq}$ to $\mathbb{Z}^m$ could be. This can be viewed as a generalization of estimating real numbers by rationals [8].

Theorem 5: [8] : Let $\mathbf{f}_i$, $i = 1, \ldots, m$ be a nondegenerate map from an open set $U_i \subset \mathbb{R}^{d_i}$ to $\mathbb{R}^n$ and

$$F : U_1 \times \ldots \times U_m \to \mathcal{M}_{m,n}, \quad (\mathbf{x}_1, \ldots, \mathbf{x}_m) \longmapsto \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1) \\ \vdots \\ \mathbf{f}_m(\mathbf{x}_m) \end{pmatrix}$$

where $\mathcal{M}_{m,n}$ denotes the space of $m \times n$ real matrices.

Then, for $\epsilon > 0$, the set of $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ such that for

$$\mathbf{A} = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}_1) \\ \vdots \\ \mathbf{f}_m(\mathbf{x}_m) \end{pmatrix} \tag{34}$$

there exist infinitely many $\mathbf{q} \in \mathbb{Z}^n$ with

$$\|\mathbf{Aq} - \mathbf{p}\| < \|\mathbf{q}\|^{\frac{-n}{m} - \epsilon} \quad \text{for some } \mathbf{p} \in \mathbb{Z}^m \tag{35}$$

has zero Lebesgue measure on $U_1 \times \ldots \times U_m$.

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